Quasi-coherent sheaves on Proj (Har II 5)

Let S be a graded ring. A graded S-module is an S-module M with a decomposition

$$M = \bigoplus_{d \in \mathbb{Z}} M_d$$

such that $S_d \cdot M_e \subseteq M_{d+e}$.

The twisted module $M(n)$ is M w/ grading shifted by n. i.e. the d^{th} graded component of $M(n)$ is

$$M(n)_d := M_{d+n}.$$

Morphisms in the category of graded S-modules are degree-preserving S-module homomorphisms.

Given a graded S-module M, we can construct a sheaf $\tilde{M}$ on $\operatorname{Proj} S$ by giving the sections on distinguished opens:

For $F \in S_+$ homogeneous, define

$$\tilde{M}(D_+(F)) = M_{(F)} := (M_F)_0 = \left\{ \frac{m}{F^n} \;\middle|\; \deg F^n = d,\ m \in M_d \right\}.$$

This is naturally an $S_{(F)} = \mathcal{O}_{\operatorname{Proj} S}(D_+(F))$-module.

$\tilde{M}$ is a sheaf w/ restriction maps localization, called the sheaf associated to M on $\mathrm{Proj}\, S$.

Properties of $\tilde{M}$:

Let $X = \mathrm{Proj}\, S$, M a graded S-module.

1. If $P \in X$, then $(\tilde{M})_P = M_{(P)} := (M_P)_0$
2. If $F \in S_+$ homogeneous, $\tilde{M}|_{D_+(F)} \cong \widetilde{M_{(F)}}$.
3. $\tilde{M}$ is a quasi-coherent $\mathcal{O}_X$-module (by 2.)

Question: If $M \mapsto \tilde{M}$ an equivalence of categories, analogous to the affine case? No:

Ex: Let $S = k[x, y]$, $I = (x)$, $J = (x^2, xy)$ ideals of S considered as S-modules, w/ standard grading.

Then $I = \underset{\deg 0}{0} \oplus k\langle x \rangle \oplus k\langle x^2, xy \rangle \oplus \cdots$

$J = \underset{\deg 0}{0} \oplus 0 \oplus k\langle x^2, xy \rangle \oplus \cdots$

In particular, $I_i = J_i = S_{i-1}x$ for $i \geq 2$.

Let $F \in S_+$ be homogeneous. Set $\deg F = d > 0$.

Then if $G \in I_m$, $FG \in I_{m+d} = J_{m+d}$, + we get a natural isom. Induced by the inclusion $J \subseteq I$:

$$J_{(F)} \longrightarrow I_{(F)}$$

$$\frac{H}{F^n} \longmapsto \frac{H}{F^n}$$

$$\frac{FG}{F^{n+1}} \longleftarrow\!\shortmid \frac{G}{F^n}$$

These maps are compatible w/ restriction, so this extends to an isomorphism $\tilde{I} \cong \tilde{J}$ even though $I \not\cong J$ as S-modules.

More generally, if M and N are graded S-modules with $M_{\geq d} \cong N_{\geq d}$ for some d, then $\tilde{M} \cong \tilde{N}$. (Use the same trick to get an isomorphism.)

We will see that this is also a necessary condition for $\tilde{M} \cong \tilde{N}$. i.e. we have an equivalence of categories between quasi-coherent sheaves on $\operatorname{Proj} S$ and graded S-modules up to equivalence, where M and N are equivalent if $M_{\geq d} \cong N_{\geq d}$ for some d.

First we need the following construction:

Def: Let $X = \operatorname{Proj} S$, $n \in \mathbb{Z}$. Define the sheaf

$$\mathcal{O}_X(n) := \widetilde{S(n)}$$

$$(S(n)_d := S_{n+d})$$

$\mathcal{O}_X(1)$ is called the twisting sheaf. If $\mathcal{F}$ is any sheaf of $\mathcal{O}_X$-modules, define

$$\mathcal{F}(n) := \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n).$$

Claim: If S is generated by S_1 as an S_0-algebra (as in the poly. ring for example), then $\mathcal{O}_X(n)$ is invertible (i.e. locally free of rank 1)

Pf: $\bigcup_{F \in S_1} D_+(F) = \{P \in X \mid S_1 \not\subseteq P\} = X$, since if $S_1 \subseteq P$ then $S_+ \subseteq P$.

Thus $D_+(F)$ cover X, for $F \in S_1$.

$\mathcal{O}_X(n)(D_+(F)) = S(n)_{(F)}$, an $S_{(F)}$-module.

$S(n)_{(F)}$ is the elts of degree n in S_F. So we get an isomorphism of $S_{(F)}$-modules:

$$\begin{aligned} S(n)_{(F)} &\longrightarrow S_{(F)} \\ m &\longmapsto \frac{m}{F^n} \\ F^n m' &\longleftarrow\!\shortmid\; m' \end{aligned}$$

Thus $\mathcal{O}_X(n)$ is locally free of rank 1. $\square$

Note that if M is any graded S-module, we have

$\tilde{M}(n) = \tilde{M} \otimes \mathcal{O}(n) = \tilde{M} \otimes \widetilde{S(n)} = \widetilde{M \otimes S(n)} = \widetilde{M(n)}$. In particular:

$$\mathcal{O}(n) \otimes \mathcal{O}(m) = \mathcal{O}(n+m).$$

If T is another graded ring also generated in degree 1, and $\varphi : S \to T$ a graded homomorphism.

Let $U = \{P \in \operatorname{Proj} T \mid P \not\supseteq \varphi(S_+)\}$, an open set in $\operatorname{Proj} T = Y$

Then φ induces a morphism $f : U \to \operatorname{Proj} S = X$ and $P \mapsto \varphi^{-1}(P)$

for M a graded S-module and N a graded T-module, just like in the affine case we have

$$f^*(\tilde{M}) \cong \left(\widetilde{M \otimes_S T}\right)\Big|_U \text{ and } f_*(\tilde{N}|_U) = \tilde{N}'.$$

N' N as an S-module

In particular,

$$f^*(\underbrace{\mathcal{O}_X(n)}_{\widetilde{S(n)}}) \cong \underbrace{\mathcal{O}_Y(n)}_{\widetilde{T(n)} \,=\, \widetilde{T \otimes_S S(n)} \text{ (check!)}}\Big|_U \text{ and } f_*(\mathcal{O}_Y(n)|_U) \cong (f_*\mathcal{O}_U)(n).$$

We know how to take a graded module and get a quasi-coherent sheaf on $\operatorname{Proj} S$.

Unlike in the affine case, we can't recover the module from a sheaf by taking global sections.

Ex: Let $S = k[x_0, \dots, x_n]$, $X = \operatorname{Proj} S = \mathbb{P}^n_k$. Then $\mathcal{O}_X(-1) = \widetilde{S(-1)}$.

Let $U_i = D_+(x_i)$. Then $\mathcal{O}_X(-1)(U_i) = S(-1)_{(x_i)} = \left\{ \frac{F}{x_i^{d+1}} \;\middle|\; F \in S_d \right\}$.

These cover X, so we can get global sections

$$\Gamma(X, \mathcal{O}_X(-1)) = \left\{ (s_i) \in \prod S(-1)_{(x_i)} \;\middle|\; s_i = s_j \text{ in } S(-1)_{(x_i x_j)} \right\}$$

But $\frac{F}{x_i^{d+1}} = \frac{G}{x_j^{e+1}} \Rightarrow x_i^{d+1} \mid \underset{\deg d}{F}$ and $x_j^{e+1} \mid \underset{\deg e}{G} \Rightarrow F = G = 0$.

So $\Gamma(X, \mathcal{O}_X(-1)) = 0 \neq S(-1)$.

Note that for $\mathcal{O}_X(n)$ with $n \geq 0$, we do have global sections: homogeneous polynomials of deg n, i.e. S_n.

More generally, we can construct a graded S-module from any sheaf by summing over every twist:

Def: $X = \operatorname{Proj} S$. Let $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-modules. The graded S-module associated to $\mathcal{F}$ is

$$\Gamma_*(\mathcal{F}) = \bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{F}(n)).$$

If $s \in S_d$ and $t \in \Gamma(X, \mathcal{F}(n))$, define

$$st := s \otimes t \in \Gamma\left(X, \mathcal{F}(n) \otimes \mathcal{O}_X(d)\right) = \Gamma\left(\mathcal{F}(n+d)\right).$$

For S a polynomial ring, this construction recovers S from the structure sheaf:

Prop: A a ring, $S = A[x_0, \ldots, x_r]$, $r \geq 1$ and $X = \operatorname{Proj} S = \mathbb{P}^r_A$. Then $\Gamma_*(\mathcal{O}_X) \cong S$.

Pf: Similar to example above. Cover X with $D_+(x_i)$ to compute global sections. (See Har for details.) $\square$

Note: For S an arbitrary graded ring, we don't necessarily have $\Gamma_*(\mathcal{O}_X) = S$.

If $\mathcal{F}$ is quasi-coherent, this construction gives us a one-sided inverse for $\sim$. Precisely:

Theorem: S a graded ring, finitely generated in S_1 as an S_0-algebra, i.e. $S = S_0[x_1, \ldots, x_n]/I$, $X = \operatorname{Proj} S$, $\mathcal{F}$ quasi-coherent on X. Then there's an isomorphism

$$\beta : \widetilde{\Gamma_*(\mathcal{F})} \longrightarrow \mathcal{F}.$$

As we saw, many modules may give the same sheaf, but $\Gamma_*(\mathcal{F})$ is the "biggest."

Up to equivalence of modules, this gives a one-to-one correspondence

$$\{S\text{-modules}/_{\text{equiv.}}\} \longleftrightarrow \{\text{quasi-coherent sheaves on } \mathrm{Proj}\, S\}$$

To prove the theorem, we need the following useful lemma, which is a generalization of one we had on Spec, and the proof is very similar, so we leave it out.

Lemma: X a scheme, $\mathcal{L}$ an invertible sheaf, $f \in \Gamma(X, \mathcal{L})$,

$$X_f = \{x \in X \mid f_x \notin \underset{\substack{\uparrow \\ \text{max'l ideal} \\ \text{in } \mathcal{O}_{x,X}}}{m_x} \mathcal{L}_x\} \underset{\text{open}}{\subseteq} X,$$

and $\mathcal{F}$ quasi-coherent.

a.) If X is quasi-compact, $s \in \Gamma(X, \mathcal{F})$ s.t. $s|_{X_f} = 0$, then $f^n s = 0$ in $\Gamma(X, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ for some $n > 0$.

b.) If $X = \bigcup U_i$, U_i open affine, $\mathcal{L}|_{U_i}$ free and $U_i \cap U_j$ quasi-compact, then given $t \in \Gamma(X_f, \mathcal{F})$, there is $n > 0$, s.t. $f^n t \in \Gamma(X_f, \mathcal{F} \otimes \mathcal{L}^{\otimes n})$ extends to a global section of $\mathcal{F} \otimes \mathcal{L}^{\otimes n}$.

(The hypotheses of a.) and b.) are satisfied if X is Noetherian or if X is quasi-compact and separated.)

Pf of Theorem: First we define $\beta : \widetilde{\Gamma_*(\mathcal{F})} \to \mathcal{F}$, by defining it on $D_+(f)$. ($f \in S_1$) Since $D_+(f)$ is affine and the sheaves are quasi-coh., it suffices to give module maps

$$\widetilde{\Gamma_*(\mathcal{F})}\left(D_+(f)\right) \longrightarrow \mathcal{F}\left(D_+(f)\right).$$

That is, we want to know the image of $\frac{m}{f^d}$, where $m \in \Gamma(X, \mathcal{F}(d))$.

$\frac{1}{f^d}$ is a section of $\mathcal{O}_X(-d)$ over $D_+(f)$, so set

$$\beta\left(\frac{m}{f^d}\right) = m \otimes \frac{1}{f^d} \in \Gamma\big(D_+(f), \underbrace{\mathcal{F}(d) \otimes \mathcal{O}_X(-d)}_{= \mathcal{F}}\big).$$

To show β is an isomorphism, first note that we can find $f_0, \ldots, f_r \in S_1$ s.t. X is covered by $D_+(f_i)$, and set $\mathcal{L} = \mathcal{O}_X(1)$. $\mathcal{L}$ is free on each $D_+(f_i)$, then we apply the lemma and get

$$\Gamma_*(\mathcal{F})_{(f)} \cong \mathcal{F}(D_+(f))$$

(Exercise! a.) injectivity
b.) surj.)

Since $\mathcal{F}$ is quasi-coherent, this implies the restrictions of the sheaves to $D_+(f)$ are isomorphic. $\square$

This has the following nice application:

Cor: Let A be a ring.

a.) If Y in $\mathbb{P}^r_A = X$ a closed subscheme, then there's a homogeneous ideal $I \subseteq S = A[x_0, \dots x_r]$ s.t. Y is the closed subsch. determined by $\operatorname{Proj}(S/I) \to X$.

b.) A scheme Y over $\operatorname{Spec} A$ is projective (i.e. closed in $\mathbb{P}^n_A$)
$\iff$ $Y \cong \operatorname{Proj} S$ for some S s.t. $S_0 = A$ and S is f.g. by S_1 as an S_0-algebra. (i.e. $S \cong A[x_1, \dots, x_n]/I$)

Pf a.) Let $\mathcal{I}_Y \subseteq \mathcal{O}_X$ the ideal sheaf of Y. First note that tensoring by $\mathcal{O}_X(n)$ is exact since it is free on some open cover, and free modules are flat.
In particular,

$$\mathcal{I}_Y(d) \subseteq \mathcal{O}_X(d).$$

Taking global sections is left exact. Thus

$$\Gamma(X, \mathcal{I}_Y(d)) \subseteq \Gamma(X, \mathcal{O}_X(d))$$

$\Rightarrow$ $\Gamma_*(\mathcal{I}_Y) \subseteq \Gamma_*(\mathcal{O}_X) = S$ (by prop above).

Thus $I = \Gamma_*(\mathcal{I}_Y)$ is a homogeneous ideal of S, which determines some closed subscheme of X whose ideal sheaf is $\tilde{I}$ (do you see why?)

By the Theorem, $\tilde{I} \cong \mathcal{I}_Y$, so Y is the subscheme determined by I.

b.) If Y is a closed subsch. of $\mathbb{P}^n_A$, then by a.),

$Y \cong \operatorname{Proj}(S/I)$, for some $I \subsetneq S_+$ so that $(S/I)_0 = A$.

(see ex 3.12)

Conversely $S = A[x_0, \dots, x_n]/I$, so Y is a closed subsch. of $\mathbb{P}^n_A$. $\square$